Differential Geometry Chapter 3

Curves

Given a curve $\alpha : I \to \mathbb{R}^n$, we will misuse notation, calling both the function and the image of the function the curve.

Definition 1 *The* ***speed function*** *of* α *is*

$$v(t) = \|\alpha'(t)\| = \left(\sum_{i=1}^{n} \alpha_i'(t)^2\right)^{1/2},$$

and

$$\int_a^b \|\alpha'(t)\| \, dt$$

is the ***arc-length*** *of* α *from* a *to* b*.*

Example 2 *For the helix* $\alpha(t) = (a\cos t, \, a\sin t, \, bt)$*, with* $a > 0$ *and* $b \neq 0$*, we have*

$$\alpha'(t) = (-a\sin t, \, a\cos t, \, b)_{\alpha(t)} .$$

Then

$$v(t) = \sqrt{a^2 + b^2}$$

and the arc-length from $t = t_0$ *to* t_1 *is* $\sqrt{a^2 + b^2}\,(t_1 - t_0)$*.*

Definition 3 *Let* I, J *be open intervals of* $\mathbb{R}$ *and* $\alpha : I \to \mathbb{R}^n$ *a curve. If* $h : J \to I$ *is a differentiable function, then the* ***reparametrization of*** α ***by*** h *is the curve* $\beta = \alpha \circ h : J \to \mathbb{R}^n$*.*

We stated the Chain Rule earlier for scalar-valued functions. Thus the following result follows immediately for the component functions $\beta^i = \alpha^i \circ h$. The stated result simply combines all these results.

Lemma 4

$$\beta'(t) = \alpha'(h(t))\, h'(t) .$$

Definition 5 *Let* $\alpha : I \to \mathbb{R}^n$ *be a curve. Then*
i. α *is a* ***regular curve*** *if* $\alpha'(t) \neq \mathbf{0}$ *for all* $t \in I$*,*
ii. α *is a* ***unit speed curve*** *if* $v(t) = 1$ *for all* $t \in I$*.*

Theorem 6 *If α is a regular curve in $\mathbb{R}^n$ then there exists a reparametrization β of α such that β has unit speed.*

Proof Choose $a \in I$ and let

$$s(t) = \int_a^t \|\alpha'(u)\| \, du,$$

the arc-length function. Since α is differentiable (i.e. in C^∞) the integrand $\|\alpha'(u)\|$ is a continuous. So the Fundamental Theorem of Calculus says that $s(t)$ is differentiable (and thus continuous) with $s'(t) = \|\alpha'(t)\|$ for all $t \in I$.

Since $\|\alpha'(u)\| > 0$ the integral $s(t)$ is a strictly increasing differentiable function. Thus the inverse function theorem states that $\operatorname{Im} s$ is an open interval in $\mathbb{R}$, J say, and s has an inverse, i.e. there exists a differentiable, strictly increasing $f : J \to I$ such that $f(s(t)) = t$ so all $t \in I$.

Let $\beta(x) = \alpha(f(x))$ for $x \in J$. Then $\beta'(x) = \alpha'(f(x))\, f'(x)$ and so

$$\begin{aligned} \|\beta'(x)\| &= \|\alpha'(f(x))\|\, f'(x) \quad \text{since } f' > 0 \\ &= s'(f(x))\, f'(x) \\ &= \frac{d}{dx} s\left(f\left(x\right)\right) \quad \text{by the Chain Rule} \\ &= \frac{d}{dx} x = 1. \end{aligned}$$

■

Example Helix $\alpha(t) = (a\cos t,\, a\sin t,\, bt)$, $t \in \mathbb{R}$, when $v(t) = \left(a^2 + b^2\right)^{1/2}$, constant but not necessarily 1. Since $0 \in \mathbb{R}$ we can start the integral at 0 in $s(t) = \int_0^t v(u)\, du = \left(a^2 + b^2\right)^{1/2} t$. Let $f(x) = x\left(a^2 + b^2\right)^{-1/2}$. Then the unit speed curve is

$$\beta(s) = \left(a\cos\left(s/c\right),\, a\sin\left(s/c\right),\, bs/c\right),\ s$$

$s \in \mathbb{R}$, where $c = \left(a^2 + b^2\right)^{1/2}$.

Note that for a unit speed curve $\alpha(t)$ we have $v(t) = 1$ for all t, so $s(t) = \int_a^t \|\alpha'(u)\|\, du = t - a$. So we can replace t by s, write $\alpha(s)$ for the curve and say that it is *parametrized by the arc-length.*

Frenet Formula

Let $\beta : I \to \mathbb{R}^n$ be a unit speed curve so $\|\beta'(s)\| = 1$ for all $s \in I$.

Definition 7 *$T(s) = \beta'(s)$ is the* ***unit tangent vector field*** *on β,*
$T'(s)$ is the ***curvature vector field*** *of β,*
$k(s) = \|T'(s)\|$ is the ***curvature function*** *of β.*

Always $\kappa(s) \geq 0$ and the larger κ is, the greater the rate of change of β in the direction of T'.

Note that

$$T \bullet T = \beta' \bullet \beta' = \|\beta'(s)\|^2 = 1$$

since the curve is of unit speed. On differentiating $T \bullet T' = 0$, i.e. T' is orthogonal to T.

Assume $\kappa(s) > 0$ for all $s \in I$.

Definition 8 *The* ***principal normal vector field*** *of β is*

$$N = N(s) = \frac{T'(s)}{\kappa(s)} = \frac{\beta''(s)}{\kappa(s)}$$

and

$$B = B(s) = T(s) \times N(s)$$

is the ***binormal vector field*** *of β.*

By definition both T and N are of unit length while $T \bullet T' = 0$ means $T \bullet N = 0$ so T and N are orthogonal. thus by an earlier result $\{T, N, B\}$ is a frame at each point of β.

Definition 9 *$\{T, N, B\}$ is the* ***Frenet frame field*** *on β.*

Example 10 *Let $\alpha(t) = (4(\cos t)/5,\ 1 - \sin t,\ -3(\cos t)/5)$ for $t \in \mathbb{R}$.*

Then $\alpha'(t) = (-4(\sin t)/5,\ -\cos t,\ 3(\sin t)/5)_{\alpha(t)}$ for which $\|\alpha'(t)\| = 1$ and so we have a unit speed curve. Thus $T(t) = \alpha'(t)$.

Next $\alpha''(t) = (-4(\cos t)/5,\ \sin t,\ 3(\cos t)/5)_{\alpha(t)}$ and again $\|\alpha''(t)\| = 1$. Thus $\kappa(t) = 1$ for all t and $N(t) = \alpha''(t)$.

Finally

$$\begin{aligned} B(t) &= T(t) \times N(t) \\ &= \left(-\frac{3}{2}\cos^2 t - \frac{3}{5}\sin^2 t,\ -\frac{12}{5}\cos t \sin t + \frac{12}{5}\sin t \cos t,\ -\frac{4}{5}\sin^2 t - \frac{4}{5}\cos^2 t\right)_{\alpha(t)} \\ &= \left(-\frac{3}{5},\ 0,\ -\frac{4}{5}\right)_{\alpha(t)}. \end{aligned}$$

Note that in this example the binormal vector is independent of the point of intersection. This is not such a surprise. The curve in question is the intersection of the sphere $(y-1)^2 + x^2 + z^2 = 1$ with the plane $3x + 4z = 0$. The binormal vector will be orthogonal to this plane, as is the vector $(-3, 0, -4)/5$.

Question How does the Frenet Frame $\{T(s), N(s), B(s)\}$ change as s changes?

Consider first $B'(s)$. Since B is of unit length $B \bullet B = 1$ and so, on differentiating, $B' \bullet B = 0$.

Also, since $\{T, N, B\}$ is a frame we have $B \bullet T = 0$, and so, on again differentiating, $B' \bullet T + B \bullet T' = 0$. But $B \bullet T' = \kappa B \bullet N = 0$ hence $B' \bullet T = 0$.

Thus, since $\{T, N, B\}$ is a frame,

$$B' = (B' \bullet T)\, T + (B' \bullet N)\, N + (B' \bullet B)\, B = (B' \bullet N)\, N.$$

Definition 11 *Define* $\tau : I \to \mathbb{R}$ *by* $B'(s) = -\tau(s)\, N(s)$, *the torsion function of* β . *Note the* $-ve$ *sign.*

Example As noted before $\beta(s) = (a\cos(s/c), a\sin(s/c), bs/c)$, $s \in \mathbb{R}$, where $c = (a^2+b^2)^{1/2}$ is a unit speed curve. Assume $a > 0$.

$$\begin{aligned} \beta'(s) &= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)_{\beta(s)} = T(s), \\ T'(s) &= \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right)_{\beta(s)} \\ &= \frac{a}{c^2}\left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)_{\beta(s)}. \end{aligned}$$

So $\kappa(s) = a/c^2$. So $N(s) = (-\cos(s/c), -\sin(s/c), 0)_{\beta(s)}$ and then

$$\begin{aligned} B(s) &= T(s) \times N(s) \\ &= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)_{\beta(s)} \times \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)_{\beta(s)} \\ &= \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right)_{\beta(s)}. \end{aligned}$$

Finally,

$$B'(s) = \left(\frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right)_{\beta(s)} = -\frac{b}{c^2}N(s).$$

Therefore $\tau(s) = b/c^2$. ■

The important observation to take away from this example is that for a helix both the curvature and torsion are constant.

Theorem 12 ***Frenet Formula*** *Let β be a unit speed curve with $\kappa(s) > 0$ for all $s \in I$. Then*

$$\begin{aligned} T'(s) &= & \kappa(s)\,N(s)\,, & \\ N'(s) &= -\kappa(s)\,T(s) & & +\tau(s)\,B(s)\,, \\ B'(s) &= & -\tau(s)\,N(s)\,. & \end{aligned}$$

Proof Only the second result here is new. Again since $\{T, N, B\}$ is a frame,

$$N' = (N' \bullet T)\,T + (N' \bullet N)\,N + (N' \bullet B)\,B.$$

From $N \bullet N = 1$ we have $N' \bullet N = 0$.

From $N \bullet T = 0$ we have $N' \bullet T + N \bullet T' = 0$, i.e. $N' \bullet T + N \bullet (\kappa N) = 0$. Thus $N' \bullet T = -\kappa$.

Similarly, from $N \bullet B = 0$ we have $N' \bullet B + N \bullet B' = 0$, i.e. $N' \bullet B - N \bullet (\tau N) = 0$. Thus $N' \bullet B = \tau$.

Combining we get the required result. ■

The plane containing T & B is the *rectifying plane*, that containing N & B the *normal* plane and that containing T & N the *osculating* plane.

Question what do κ and τ represent?

Consider the unit speed curve $\beta(s)$. Taylor's expansion states that, for s sufficiently small,

$$\beta(s) = \beta(0) + \beta'(0)\,s + \beta''(0)\,\frac{s^2}{2} + \beta'''(0)\,\frac{s^3}{3!} + \ldots.$$

Here $\beta'(0) = T(0)$ and $\beta''(0) = T'(0) = \kappa(0)\,N(0)$. Also

$$\begin{aligned}\beta'''(0) &= \frac{d}{ds}\left(\kappa(s)\,N(s)\right)\bigg|_{s=0} = \kappa'(0)\,N(0) + \kappa(0)\,N'(0) \\ &= \kappa'(0)\,N(0) + \kappa(0)\left(-\kappa(0)\,T(0) + \tau(0)\,B(0)\right)\end{aligned}$$

Substituting back,

$$\beta(s) \approx \beta(0) + \left(s - \frac{\kappa^2(0)}{6}s^3\right)T(0) + \left(\kappa(0)\,\frac{s^2}{2} + \kappa'(0)\,\frac{s^3}{6}\right)N(0) + \kappa(0)\,\tau(0)\,\frac{s^3}{6}B(0)\,. \tag{1}$$

So a first approximation to $\beta(s)$ is the tangent line $\beta(0) + sT(0)$. The second is the parabola

$$\beta(0) + sT(0) + \kappa(0)\,\frac{s^2}{2}N(0)\,. \tag{2}$$

Thus $\kappa(0)$ controls how fast the curve diverges from the straight line in the direction of $N(0)$ (how much it *bends*). Note that as s varies, the curve (2) lies in the plane $\beta(0) + \text{span}\,\{T(0)\,, N(0)\}$, the osculating plane mentioned before. We say that the osculating plane is the best approximating plane to β at $\beta(0)$.

If we had more time we would talk about the osculating circle, and the evolute and involute curves. But we don't!

The third approximation is the cubic (1). Hence $\tau(0)$ controls how fast the curve leaves the $\{T(0)\,, N(0)\}$ plane (or how much the curve *twists*.).

Question If $\tau(s) = 0$ for all s does the curve remain in the $\{T(0)\,, N(0)\}$ plane? (If so we say, unsurprisingly, that the curve is **planar**.)

Lemma 13 *Let β be a unit speed curve with $\kappa(s) > 0$ for all $s \in I$. Then β is planar iff $\tau(s) = 0$ for all $s \in I$.*

Proof ($\Longrightarrow$) If β is planar then there exist points $\mathbf{p}$ and normal vector $\mathbf{n}$, such that $(\beta(s) - \mathbf{p}) \bullet \mathbf{n} = 0$. Differentiating two times

$$\beta'(s) \bullet \mathbf{n} = \beta''(s) \bullet \mathbf{n} = 0, \quad \text{i.e.} \quad T(s) \bullet \mathbf{n} = \kappa(s)\, N(s) \bullet \mathbf{n} = 0$$

for all s. This means that $\mathbf{n}$ is orthogonal to both $T(s)$ and $N(s)$ for all s. Yet $B(s)$ is also orthogonal to both $T(s)$ and $N(s)$ and so $B(s) = \pm\mathbf{n}/\left\|\mathbf{n}\right\|$ for all s. (This steps uses the fact that we have only 3 dimensions.) Therefore $B'(s) = 0$, i.e. $\tau(s) = 0$ for all s.

($\Longleftarrow$) Assuming $\tau(s) = 0$ for all s means $B'(s) = 0$, i.e. $B(s)$ is constant for all s. Claim $(\beta(s) - \beta(0)) \bullet B(s) = 0$, i.e. β is planar.

Let $f(s) = (\beta(s) - \beta(0)) \bullet B(s)$. Then $f'(s) = \beta'(s) \bullet B(s) = T(s) \bullet B(s) = 0$. So $f(s)$ is constant. Yet $f\,(0) = 0$ so $f(s) = 0$ for all s as claimed.

■

Example 14 *In the earlier example of* $\alpha(t) = (4\,(\cos t)\,/5,\ 1 - \sin t,\ -3\,(\cos t)\,/5)$ *for* $t \in \mathbb{R}$ *we found* $B(t) = \left(-\frac{3}{5},\ 0,\ -\frac{4}{5}\right)_{\alpha(t)}$. *Thus* $B'(t) = (0,\ 0,\ 0)_{\alpha(t)}$ *in which case* $\tau(t) = 0$ *for all* t *and the curve is planar.*

Further, from the proof of the lemma, the curve lies in the plane $(\mathbf{x} - \alpha\,(0)) \bullet B\,(t) = 0$. That is,

$$\left(x - \frac{4}{5},\ y - 1,\ z + \frac{3}{5}\right) \bullet \left(-\frac{3}{5},\ 0,\ -\frac{4}{5}\right) = 0,$$

or $3x + 4z = 0$.

In this example we also found that $\kappa(t) = 1$ for all t. This is a special case of

Lemma 15 *If* $\tau \equiv 0$ *and* $\kappa(s)$ *is constant then* β *is part of a circle.*

Proof By Lemma 13, $\tau \equiv 0$ means that β is planar. Consider the curve

$$\gamma(s) = \beta(s) + \frac{1}{\kappa} N(s)\,, \tag{3}$$

where $\kappa = \kappa(s)$. Then, since κ is constant,

$$\begin{aligned} \gamma'(s) &= \beta'(s) + \frac{1}{k} N'(s) \\ &= T(s) + \frac{1}{\kappa}\left(-\kappa(s)\, T(s) + \tau(s)\, B(s)\right) \\ &= 0. \end{aligned}$$

Thus $\gamma\,(s)$ is constant, i.e. equal to some $\mathbf{p} \in \mathbb{R}^n$. Then, rearranging (3) and taking norms,

$$\|\beta(s) - \mathbf{p}\| = \frac{1}{\kappa} \|N(s)\| = \frac{1}{\kappa},$$

i.e. $\beta(s)$ lies on a circle, centre $\mathbf{p}$, radius $1/\kappa$. ■

Example 16 *In the earlier example of* $\alpha(t) = (4\,(\cos t)\,/5,\, 1 - \sin t,\, -3\,(\cos t)\,/5)$ *for* $t \in \mathbb{R}$ *we found that* $N(t) = (-4\,(\cos t)\,/5,\, \sin t,\, 3\,(\cos t)\,/5)_{\alpha(t)}$ *and* $\kappa(t) = 1$ *for all* t*. Thus* $\alpha(t)$ *lies on the circle or radius 1, centre*

$$\alpha(0) + \frac{1}{\kappa} N(0) = \left(\frac{4}{5},\, 1,\, -\frac{3}{5}\right) + \left(-\frac{4}{5},\, 0,\, \frac{3}{5}\right) = (0,\, 1,\, 0)\,.$$

Instead of lying in a circle what if the curve lies in the surface of a sphere?

Lemma 17 *If the image of the unit speed* $\alpha : I \to \mathbb{R}^n$ *lies within the surface of a sphere, then* $\kappa(t) \neq 0$ *and*

$$\rho^2 + \left(\rho' \sigma\right)^2 = r^2,$$

where r *is the radius of the sphere,* $\rho\,(t) = 1/\kappa(t)$ *and* $\sigma(t) = 1/\tau(t)\,.$

Proof That α lies on the surface of a sphere means there is a point $\mathbf{c} \in \mathbb{R}^n$ and radius $r > 0$ such that

$$(\alpha(t) - \mathbf{c}) \bullet (\alpha(t) - \mathbf{c}) = r^2$$

for all $t \in I$. For simplicity I drop the dependency on t from my expressions.

The first differentiation gives $\alpha' \bullet (\alpha - \mathbf{c}) = 0$, i.e.

$$T \bullet (\alpha - \mathbf{c}) = 0. \tag{4}$$

Differentiate again, $T' \bullet (\alpha - \mathbf{c}) + T \bullet T = 0$ i.e. $\kappa N \bullet (\alpha - \mathbf{c}) = -1$. Thus $\kappa \neq 0$ and

$$N \bullet (\alpha - \mathbf{c}) = -\rho. \tag{5}$$

Differentiate again, $N' \bullet (\alpha - \mathbf{c}) + N \bullet \alpha' = -\rho'$. But $N \bullet \alpha' = N \bullet T = 0$ while $N' = -\kappa T + \tau B$. Thus $(-\kappa T + \tau B) \bullet (\alpha - \mathbf{c}) = -\rho'$. Yet, from above, $T \bullet (\alpha - \mathbf{c}) = 0$, so $\tau B \bullet (\alpha - \mathbf{c}) = -\rho'$, i.e.

$$B \bullet (\alpha - \mathbf{c}) = -\sigma \rho'. \tag{6}$$

Since $\{T, N, B\}$ is a frame,

$$\begin{aligned} \alpha - \mathbf{c} &= ((\alpha - \mathbf{c}) \bullet T)\, T + ((\alpha - \mathbf{c}) \bullet N)\, N + ((\alpha - \mathbf{c}) \bullet B)\, B \\ &= -\rho N - \sigma \rho' B. \end{aligned}$$

by $(4), (5)$ and (6). Returning to the definition of a sphere

$$\begin{aligned} r^2 &= (-\rho N - \sigma \rho' B) \bullet (-\rho N - \sigma \rho' B) \\ &= \rho^2 N \bullet N + (\sigma \rho')^2\, B \bullet B \\ &= \rho^2 + (\sigma \rho')^2 . \end{aligned}$$

■

Why are κ and τ of interest?

There are perhaps many answers to this question but I'm interested in the fact that a "unit speed curve is uniquely determined by the pair of functions $(\kappa(t), \tau(t))$, up to position in $\mathbb{R}^3$".

The map between the same object in different positions is the following.

Definition 18 *A map* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *is an* ***isometry*** *if* $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

There are four basic isometries, reflection; glide reflection; rotation and translation Isometries are given by an affine map $\mathbf{x} \mapsto \mathbf{a} + A\mathbf{x}$, with $\mathbf{a} \in \mathbb{R}^3$ and 3×3 orthogonal matrix A, so $A^T A = I_3$. The uniqueness result is

Theorem 19 *If $\alpha, \beta : I \to \mathbb{R}^3$ are unit speed curves with $(\kappa_\alpha(t)\,, \tau_\alpha(t)) = (\kappa_\beta(t)\,, \tau_\beta(t))$ for all $t \in I$ then there exists an isometry $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\beta = \mathbf{F} \circ \alpha$.*

Proof not given, but the isometry is constructed by mapping the Frenet Frame on α at time t to the Frenet Frame on β at the same time. ■

Example If a unit speed curve $\alpha : I \to \mathbb{R}^3$ has constant curvature and torsion then it is a helix of the form $\beta(s) = (a\cos(s/c)\,, a\sin(s/c)\,, bs/c)$ for some a, $b \in \mathbb{R}$.

Another fundamental result is one of existence .

Theorem 20 *Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc-length, $\kappa(s)$ is the curvature and $\tau(s)$ the torsion of α.*

Proof not given, but involves the existence and uniqueness of ordinary differential equations. ■

Arbitrary Speed Curves

If $\alpha(t)$ is not of unit speed we can find the arc-length parameter $s(t)$ and $\beta(s)$ a unit speed curve satisfying $\beta(s(t)) = \alpha(t)$.

Calculate $T_\beta(s)\,, N_\beta(s)\,, B_\beta(s)\,, \kappa_\beta(s)$ and $\tau_\beta(s)$ for $\beta\,(s)$. Then $\{T_\beta(s)\,, N_\beta(s)\,, B_\beta(s)\}$ is a frame for β for all s.

Write $T(t) = T_\beta(s(t))\,,\ N(t) = N_\beta(s(t))\,,\ B(t) = B_\beta(s(t))\,,\ \kappa(t) = \kappa_\beta(s(t))$ and $\tau(t) = \tau_\beta(s(t))$. Then $\{T(t)\,, N(t)\,, B(t)\}$ is a frame for α all t.

To see how this frame for α transforms as t varies,

$$\frac{d}{dt}T(t) = \frac{d}{ds}T_\beta(s)\,\frac{d}{dt}s(t) = \kappa_\beta(s)\,N_\beta(s)\,v(t) = \kappa(t)\,N(t)\,v(t)\,,$$

having used the Frenet formula for unit speed curves, Theorem 12. And

$$\begin{aligned}\frac{d}{dt}N(t) &= \frac{d}{ds}N_\beta(s)\,\frac{d}{dt}s(t) = (-\kappa_\beta(s)\,T_\beta(s) + \tau_\beta(s)\,B_\beta(s))\,v(t)\\ &= (-\kappa(t)\,T(t) + \tau(t)\,B(t))\,v(t)\,.\end{aligned}$$

Finally

$$\frac{d}{dt}B(t) - \tau(t)\, N(t)\, v(t)\,.$$

So, for an arbitrary speed curve the Frenet formula become

$$\begin{array}{rlll} T'(t) = & & \kappa(t)\, v(t)\, N(t)\,, & \\ N'(t) = & -\kappa(t)\, v(t)\, T(t) & & +\tau(t)\, v(t)\, B(t)\,, \\ B'(t) = & & -\tau(t)\, v(t)\, N(t)\,. & \end{array}$$

Calculations

When it comes to calculations we can quickly differentiate the given $\alpha(t)$ but how to use these derivatives to calculate the T, N, B, κ and τ?

First

$$\alpha'(t) = \frac{d}{ds}\beta(s)\, \frac{d}{dt}s(t) = T_\beta(s(t))\, v(t) = T(t)\, v(t)\,.$$

Yet T is of unit length so $T = \alpha'(t)\,/\,\|\alpha'(t)\|$.

Continue differentiating,

$$\begin{aligned} \alpha''(t) &= \frac{d}{dt}T_\beta(s(t))\, v(t) + T(t)\, v'(t) \\ &= \kappa_\beta(s(t))\, N_\beta(s(t))\, v^2(t) + T(t)\, v'(t) \\ &= \kappa(t)\, N(t)\, v^2(t) + T(t)\, v'(t)\,. \end{aligned}$$

This shows that the acceleration of $\alpha(t)$ has a tangential component, the $T(t)\, v'(t)$ term, and a normal component proportional to the square of velocity and to the curvature of the curve..

Next, again dropping the dependency on t for ease of notation,

$$\alpha' \times \alpha'' = (Tv) \times \left(\kappa N v^2 + T v'\right) = \kappa v^3 B,$$

since $B = T \times N$ and $T \times T = 0$. Yet B is of unit length so $B = \alpha' \times \alpha'' / \|\alpha' \times \alpha''\|$. And for the same reason, $\kappa v^3 = \|\alpha' \times \alpha''\|$, and thus

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}.$$

Find N from $N = B \times T = (\alpha' \times \alpha'') \times \alpha' / \|(\alpha' \times \alpha'') \times \alpha'\|$.

For the final derivative we have

$$\begin{aligned}\alpha''' &= \left(\kappa v^2\right)' N + \kappa v^2 N' + v'' T + v' T' \\ &= \left(\kappa v^2\right)' N + \kappa v^2 \left(-\kappa v T + \tau v B\right) + v'' T + v' \kappa v N \\ &= \left(v'' - \kappa^2 v^3\right) T + \left(\left(\kappa v^2\right)' + v' \kappa v\right) N + \kappa \tau v^3 B.\end{aligned}$$

We only need to know the coefficient of B here, since

$$(\alpha' \times \alpha'') \bullet \alpha''' = \kappa^2 \tau v^6.$$

Hence

$$\tau = \frac{\|(\alpha' \times \alpha'') \bullet \alpha'''\|}{\|\alpha' \times \alpha''\|^2},$$

since, from earlier, $\kappa v^3 = \|\alpha' \times \alpha''\|$.

Example 21 *Find T, N, B, κ and τ for the curve*

$$\alpha(t) = (a \cos t,\ a \sin t,\ d \sin t),$$

$t \in \mathbb{R}$.

Solution First $\alpha'(t) = (-a \sin t, a \cos t, d \cos t)_{\alpha(t)}$ so $\|\alpha'(t)\| = \left(a^2 + d^2 \cos^2 t\right)^{1/2}$.

Continuing,

$$\begin{aligned}\alpha''(t) &= (-a \cos t,\ -a \sin t,\ -d \sin t)_{\alpha(t)} \\ \alpha'''(t) &= (a \sin t,\ -a \cos t,\ -d \cos t)_{\alpha(t)}.\end{aligned}$$

Then $\alpha' \times \alpha'' = (0, -ad, a^2)_{\alpha(t)}$ and $\|\alpha' \times \alpha''\| = a\left(a^2 + d^2\right)^{1/2}$.

For N we need

$$\begin{aligned}(\alpha' \times \alpha'') \times \alpha' &= \left(0, -ad, a^2\right)_{\alpha(t)} \times (-a \sin t, a \cos t, d \cos t)_{\alpha(t)} \\ &= \left(\left(-ad^2 - a^3\right) \cos t, -a^3 \sin t, -a^2 d \sin t\right)_{\alpha(t)}.\end{aligned}$$

Then

$$\begin{aligned}\|(\alpha' \times \alpha'') \times \alpha'\|^2 &= \left(ad^2 + a^3\right)^2 \cos^2 t + \left(a^6 + a^4 d^2\right) \sin^2 t \\ &= a^2 \left(a^2 + d^2\right) \left(\left(a^2 + d^2\right) \cos^2 t + a^2 \sin^2 t\right) \\ &= a^2 \left(a^2 + d^2\right) \left(a^2 + d^2 \cos^2 t\right).\end{aligned}$$

Next, $(\alpha' \times \alpha'') \bullet \alpha''' = a^2 d \cos t - a^2 d \cos t = 0$.

Putting these results together,

$$T = \frac{\alpha'(t)}{\|\alpha'(t)\|} = \frac{(-a \sin t,\ a \cos t,\ d \cos t)_{\alpha(t)}}{(a^2 + d^2 \cos^2 t)^{1/2}},$$

$$N = \frac{(\alpha' \times \alpha'') \times \alpha'}{\|(\alpha' \times \alpha'') \times \alpha'\|} = \frac{((-ad^2 - a^3) \cos t,\ -a^3 \sin t,\ -a^2 d \sin t)_{\alpha(t)}}{a(a^2 + d^2)^{1/2} (a^2 + d^2 \cos^2 t)^{1/2}},$$

$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} = \frac{(0,\ -ad,\ a^2)_{\alpha(t)}}{a(a^2 + d^2)^{1/2}},$$

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{a(a^2 + d^2)^{1/2}}{(a^2 + d^2 \cos^2 t)^{3/2}},$$

and

$$\tau = \frac{\|(\alpha' \times \alpha'') \bullet \alpha'''\|}{\|\alpha' \times \alpha''\|^2} = 0,$$

i.e. the curve is planar. (By observation it lies in the plane $dy - az = 0$.) Finally